

# LAGUERRE-GAUSSIAN MODES AND THE WIGNER TRANSFORM

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**ABSTRACT.** Recent developments in laser physics have called renewed attention to Laguerre-Gaussian (LG) beams of paraxial light. In this paper we consider the corresponding LG modes for the two-dimensional harmonic oscillator, which appear in the transversal plane at the laser beam's waist. We see how they arise as Wigner transforms of Hermite-Gaussian modes, and we proceed to find a closed form for their own Wigner transforms, providing an alternative to the methods of Simon and Agarwal. Our main observation is that the Wigner transform intertwines the creation and annihilation operators for the two classes of modes.

**Keywords:** Laguerre-Gaussian modes, Laguerre-Gaussian beams, Wigner transform, optical angular momentum, orbital angular momentum

## 1. INTRODUCTION

In the early 1990s it was observed that a Laguerre-Gaussian (LG) beam of paraxial light has a well-defined orbital angular momentum and that such a beam may be created from a Hermite-Gaussian (HG) beam by way of an astigmatic optical system [1]. This discovery was soon to find applications, for example, in biology (for “optical tweezers”) and in the study of quantum entanglement [2]. Additionally, recent experiments suggest that beams with orbital angular momentum might be generated *in situ*, in free electron lasers [3].

In this paper we consider the corresponding LG modes for the two-dimensional harmonic oscillator, which appear in the transversal plane at the laser beam's waist. Indeed, one may use the operator algebra of the harmonic oscillator to study the analytical forms of the HG and LG beams [4], [5]. Here we demonstrate that the LG modes arise as Wigner transforms of HG modes, and we proceed to find their own Wigner transforms. Our methods provide an alternative to those of Gase [6] and those of Simon and Agarwal [7]; in particular, Simon and Agarwal were first to discover the closed form expression that we rederive here, using our new point of view.

We define the  $d$ -dimensional Wigner transform as

$$W_d(f, g)(\vec{x}, \vec{\xi}) = (2\pi)^{-\frac{d}{2}} \int e^{i\vec{p} \cdot \vec{\xi}} \overline{f\left(\frac{\vec{x} + \vec{p}}{\sqrt{2}}\right)} g\left(\frac{\vec{x} - \vec{p}}{\sqrt{2}}\right) d\vec{p}$$

for functions  $f, g$  of  $d$  variables. If  $d = 1$ , we omit the subscript. The Wigner transform is often restricted to the case when  $f \equiv g$ , and then one writes  $W_d(f) := W_d(f, f)$ . More generally, we define the extended Wigner transform, of a function  $F$  of  $2d$  variables, as

$$\tilde{W}_d(F)(\vec{x}, \vec{\xi}) = (2\pi)^{-\frac{d}{2}} \int e^{i\vec{p} \cdot \vec{\xi}} F\left(\frac{\vec{x} + \vec{p}}{\sqrt{2}}, \frac{\vec{x} - \vec{p}}{\sqrt{2}}\right) d\vec{p}.$$

This more general transform is fundamental for the study of LG modes; this is our main insight and provides the central theme for our paper.

Our main results are summarized in the following theorem.

**Theorem 1.** *Let  $h_j$  denote the  $j^{\text{th}}$  Hermite function, so that  $h_{jk}(x, y) = h_j(x)h_k(y)$  are the HG modes.*

- (a) *The extended Wigner transform intertwines the creation and annihilation operators of the HG and LG modes.*
- (b) *The LG modes are precisely  $\tilde{W}(h_{jk})$ : the extended Wigner transforms of the HG modes.*
- (c) *The Wigner transforms of the LG modes are given by*

$$(1) \quad \begin{aligned} & W_2(\tilde{W}(h_{jk}), \tilde{W}(h_{mn}))(\vec{x}, \vec{\xi}) \\ &= \tilde{W}(h_{jm})\left(\frac{x_1 + \xi_2}{\sqrt{2}}, \frac{\xi_1 - x_2}{\sqrt{2}}\right) \tilde{W}(h_{kn})\left(\frac{x_1 - \xi_2}{\sqrt{2}}, \frac{\xi_1 + x_2}{\sqrt{2}}\right). \end{aligned}$$

We remark that part (b) of the theorem follows from part (a) and the fact that  $\tilde{W}(h_{00}) = h_{00}$ .

In the case  $(j, k) = (m, n)$ , the formula (1) was proven by Simon and Agarwal, using the metaplectic representation, and in fact their proof extends to the general case without modification [7]. The generalization should be useful in light of the relationship between the Wigner transform and the Weyl quantization of observables. Indeed, for any tempered distribution  $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$  we may define the Weyl quantization of  $\sigma$  as the [pseudodifferential] operator given by

$$(\text{Op}^W(\sigma)u)(x) = 2^{-\frac{3d}{2}}\pi^{-d} \iint \exp\left(\frac{i(x-y)\xi}{\sqrt{2}}\right) \sigma\left(\frac{x+y}{\sqrt{2}}, \xi\right) u(y) dy d\xi.$$

The normalization factor is chosen so that  $\text{Op}^W(1)$  is the identity operator. Then it is easy to check that for any Schwartz functions  $f, g \in \mathcal{S}(\mathbb{R}^d)$  we have

$$\langle f | \text{Op}^W(\sigma)g \rangle = 2^{-d}\pi^{-\frac{d}{2}} \iint \sigma(x, \xi) W_d(f, g)(x, \xi) dx d\xi.$$

For an exposition of the Weyl calculus of pseudodifferential operators, the reader may consult the book of Gerald Folland [8].

In Section 2 we review the theory of the one-dimensional harmonic oscillator. Since this subject is very well known, we will be brief, so the section is mostly a means of setting up notation for the following sections. In Section 3, we review the theory of the two-dimensional harmonic oscillator. Our primary reference for Sections 2 and 3 is the classic book of Messiah [9].

In Section 4 we write the  $\{Q\}$ -representation in terms of complex variables, which provides a clean expression for the LG modes in terms of Laguerre polynomials. The connection with LG modes of paraxial light is briefly exhibited in Section 5.

The LG modes are expressed as Wigner transforms in Section 6, and in Section 7 we calculate their own Wigner transforms.

## 2. THE ONE-DIMENSIONAL HARMONIC OSCILLATOR

We begin by setting up the notation for the one-dimensional harmonic oscillator, given by

$$H = \frac{1}{2}(P^2 + Q^2), \quad \text{where } [Q, P] = i.$$

We set

$$a = \frac{1}{\sqrt{2}}(Q + iP) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP)$$

so that

$$[a, a^\dagger] = 1 \quad \text{and} \quad H = \frac{1}{2}(aa^\dagger + a^\dagger a).$$

Moreover, we define

$$N = a^\dagger a$$

so that

$$H = N + \frac{1}{2}.$$

It is well known that  $\text{Spec}(N) = \{0, 1, 2, \dots\}$  consists of nondegenerate eigenvalues and that  $a^\dagger$  and  $a$  are the “ladder operators”: the creation and annihilation operators, respectively. The associated eigenvectors form a complete set, so we can normalize to get an orthonormal basis of eigenvectors for the observable  $N$ :

$$|0\rangle, \quad |1\rangle, \quad |2\rangle, \dots$$

corresponding to the eigenvalues

$$0, \quad 1, \quad 2, \dots$$

In the  $\{Q\}$ -Representation (that is, the Schrödinger representation where states are considered as functions of position) the ladder operators are written as

$$a = \frac{1}{\sqrt{2}} \left( x + \frac{d}{dx} \right)$$

and

$$a^\dagger = \frac{1}{\sqrt{2}} \left( x - \frac{d}{dx} \right).$$

The ground state  $h_0$  satisfies

$$\left[ \frac{d}{dx} + x \right] h_0(x) = 0,$$

so that

$$h_0(x) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}x^2}.$$

Using the ladder operators, we get the rest of the eigenvectors, which are precisely the Hermite functions:

$$\begin{aligned} h_n(x) &= \pi^{-\frac{1}{4}} (n!)^{-\frac{1}{2}} 2^{-\frac{n}{2}} \left( x - \frac{d}{dx} \right)^n e^{-\frac{1}{2}x^2} \\ &= \pi^{-\frac{1}{4}} (n!)^{-\frac{1}{2}} 2^{-\frac{n}{2}} (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} e^{-x^2} \\ &= \pi^{-\frac{1}{4}} (n!)^{-\frac{1}{2}} 2^{-\frac{n}{2}} e^{-\frac{1}{2}x^2} H_n(x) \end{aligned}$$

where the  $H_n$  are the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

### 3. THE TWO-DIMENSIONAL ISOTROPIC HARMONIC OSCILLATOR

In two dimensions, we have

$$H = \frac{1}{2}(P_1^2 + P_2^2 + Q_1^2 + Q_2^2)$$

with creation and annihilation operators inherited from the one-dimensional case:

$$\begin{aligned} a_j^\dagger &= \frac{1}{\sqrt{2}}(Q_j - iP_j), \\ a_j &= \frac{1}{\sqrt{2}}(Q_j + iP_j), \quad j = 1, 2. \end{aligned}$$

These are interpreted as creation and annihilation operators, respectively, of quanta of type  $j \in \{1, 2\}$ . They satisfy

$$\begin{aligned} [a_i, a_j] &= [a_i^\dagger, a_j^\dagger] = 0, \\ [a_i, a_j^\dagger] &= \delta_{ij}. \end{aligned}$$

And we have the corresponding “number operators”

$$N_j = a_j^\dagger a_j, \quad j = 1, 2,$$

with

$$N = N_1 + N_2$$

representing the total number.

Now the eigenvalues for

$$H = N + 1$$

are

$$\{1, 2, 3, \dots\}$$

and the eigenvalue  $j$  has degeneracy  $j$ . Hence there are many possible choices of eigenbasis. One often simply takes the basis consisting of tensor products of the one-dimensional eigenvectors, corresponding to the complete set of commuting observables  $\{N_1, N_2\}$ . The elements of this basis are given by

$$|n_1 n_2\rangle = (n_1! n_2!)^{-\frac{1}{2}} a_1^{\dagger n_1} a_2^{\dagger n_2} |00\rangle.$$

These are precisely the HG modes; in the Schrödinger representation, they are tensor products of Hermite functions.

But here we will construct another basis.

The angular momentum operator  $L$  is defined by

$$L = Q_1 P_2 - Q_2 P_1 = i(a_1 a_2^\dagger - a_1^\dagger a_2),$$

and one may check that it is a constant of motion. We will show that  $\{N, L\}$  is another complete set of commuting observables.

Let

$$A_{\pm} = \frac{1}{\sqrt{2}}(a_1 \mp ia_2)$$

and

$$A_{\pm}^{\dagger} = \frac{1}{\sqrt{2}}(a_1^{\dagger} \pm ia_2^{\dagger}).$$

Then

$$\begin{aligned} [A_r, A_s] &= [A_r^{\dagger}, A_s^{\dagger}] = 0 \quad \text{and} \\ [A_r, A_s^{\dagger}] &= \delta_{rs} \quad \text{for } r, s \in \{+, -\}. \end{aligned}$$

We think of  $A_r^{\dagger}$ ,  $A_r$  as creation and annihilation operators, respectively, of quanta of type  $r \in \{+, -\}$ , so then

$$N_r = A_r^{\dagger} A_r$$

represents the number of “ $r$  quanta”.

Hence the problem of finding eigenvectors common to  $N_+$  and  $N_-$  is formally equivalent to finding eigenvectors common to  $N_1$  and  $N_2$ . So by the usual arguments we see that

$$\text{Spec}(N_+) = \text{Spec}(N_-) = \{0, 1, 2, \dots\}$$

and that these two observables,  $N_+$  and  $N_-$ , form a complete set of commuting observables: to each  $(n_+, n_-)$  there is a common eigenvector, denoted by  $|n_+ n_- \rangle$ , that is unique to within a constant.

In fact,

$$A_+ |00\rangle = A_- |00\rangle = 0$$

and the states

$$|n_+ n_- \rangle = (n_+! n_-!)^{-\frac{1}{2}} A_+^{\dagger n_+} A_-^{\dagger n_-} |00\rangle$$

form a complete orthonormal eigenbasis common to  $N_+$  and  $N_-$ :

$$N_+ |n_+ n_- \rangle = n_+ |n_+ n_- \rangle$$

and

$$N_- |n_+ n_- \rangle = n_- |n_+ n_- \rangle.$$

We find that

$$N = N_+ + N_-$$

and that

$$L = N_+ - N_-.$$

Hence  $N$  and  $L$  form a complete set of commuting observables. Moreover,

$$[L, A_{\pm}^{\dagger}] = \pm A_{\pm}^{\dagger}$$

and

$$[L, A_{\pm}] = \mp A_{\pm},$$

so when they act upon an eigenvector of  $L$ ,  $A_+^{\dagger}$  and  $A_-$  increase  $L$  by one unit, and  $A_-^{\dagger}$  and  $A_+$  decrease  $L$  by one unit. So it is natural to consider  $N_+$  as the number of particles with positive charge,  $N_-$  as the number of particles with negative charge, and  $L$  as the total charge (to within a constant).

#### 4. THE TWO-DIMENSIONAL ISOTROPIC HARMONIC OSCILLATOR IN THE $\{Q\}$ -REPRESENTATION

In the  $\{Q\}$ -representation, we write

$$(2) \quad \begin{aligned} a_1 &= \frac{1}{\sqrt{2}} \left( x + \frac{\partial}{\partial x} \right), & a_2 &= \frac{1}{\sqrt{2}} \left( y + \frac{\partial}{\partial y} \right), \\ a_1^\dagger &= \frac{1}{\sqrt{2}} \left( x - \frac{\partial}{\partial x} \right), & a_2^\dagger &= \frac{1}{\sqrt{2}} \left( y - \frac{\partial}{\partial y} \right), \end{aligned}$$

so that we have

$$(3) \quad \begin{aligned} A_+ &= \frac{1}{2} \left( x + \frac{\partial}{\partial x} - i \left( y + \frac{\partial}{\partial y} \right) \right), \\ A_- &= \frac{1}{2} \left( x + \frac{\partial}{\partial x} + i \left( y + \frac{\partial}{\partial y} \right) \right), \\ A_+^\dagger &= \frac{1}{2} \left( x - \frac{\partial}{\partial x} + i \left( y - \frac{\partial}{\partial y} \right) \right), \\ A_-^\dagger &= \frac{1}{2} \left( x - \frac{\partial}{\partial x} - i \left( y - \frac{\partial}{\partial y} \right) \right). \end{aligned}$$

In complex notation, these take a simple form. We write  $z = x + iy$ , so that

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial y} \right).$$

Then

$$\begin{aligned} A_+ &= \frac{1}{2} \bar{z} + \frac{\partial}{\partial z}, & A_- &= \frac{1}{2} z + \frac{\partial}{\partial \bar{z}}, \\ A_+^\dagger &= \frac{1}{2} z - \frac{\partial}{\partial \bar{z}}, & A_-^\dagger &= \frac{1}{2} \bar{z} - \frac{\partial}{\partial z}. \end{aligned}$$

Moreover, we can write

$$A_+^\dagger = -e^{\frac{1}{2}z\bar{z}} \frac{\partial}{\partial \bar{z}} e^{-\frac{1}{2}z\bar{z}} \quad \text{and} \quad A_-^\dagger = -e^{\frac{1}{2}z\bar{z}} \frac{\partial}{\partial z} e^{-\frac{1}{2}z\bar{z}}.$$

The ground state is given by

$$u_0(z, \bar{z}) = \pi^{-\frac{1}{2}} e^{-\frac{1}{2}z\bar{z}} \equiv \langle z, \bar{z} | 00 \rangle,$$

and of course we get all other eigenvectors by applying the creation operators to this. These are precisely the LG modes. Explicitly,

$$\begin{aligned} \langle z, \bar{z} | n_+ n_- \rangle &= \langle z, \bar{z} | (n_+! n_-!)^{-\frac{1}{2}} A_+^{\dagger n_+} A_-^{\dagger n_-} | 00 \rangle \\ &= \pi^{-\frac{1}{2}} (n_+! n_-!)^{-\frac{1}{2}} (-1)^{n_+ + n_-} e^{\frac{1}{2}z\bar{z}} \left( \frac{\partial}{\partial \bar{z}} \right)^{n_+} \left( \frac{\partial}{\partial z} \right)^{n_-} e^{-z\bar{z}}. \end{aligned}$$

We will now formulate this in terms of Laguerre polynomials.

First suppose that  $n_+ \geq n_-$ . Then

$$\begin{aligned}\langle z, \bar{z} | n_+ n_- \rangle &= \pi^{-\frac{1}{2}} (n_+! n_-!)^{-\frac{1}{2}} (-1)^{n_+ + n_-} e^{\frac{1}{2} z \bar{z}} \left( \frac{\partial}{\partial z} \right)^{n_-} [(-z)^{n_+} e^{-z \bar{z}}] \\ &= \pi^{-\frac{1}{2}} (n_+! n_-!)^{-\frac{1}{2}} (-1)^{n_-} e^{\frac{1}{2} z \bar{z}} \bar{z}^{(n_- - n_+)} \left( \frac{\partial}{\partial(z \bar{z})} \right)^{n_-} [(z \bar{z})^{n_+} e^{-z \bar{z}}].\end{aligned}$$

We define the Laguerre polynomials by

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha})$$

for  $n \geq 0$  and  $\alpha > -1$ . Then we see that, when  $n_+ \geq n_-$ ,

$$\langle z, \bar{z} | n_+ n_- \rangle = \pi^{-\frac{1}{2}} \left( \frac{n_-!}{n_+!} \right)^{\frac{1}{2}} (-1)^{n_-} z^{(n_+ - n_-)} e^{-\frac{1}{2} z \bar{z}} L_{n_-}^{n_+ - n_-}(z \bar{z}).$$

On the other hand, if  $n_+ \leq n_-$ , similar arguments show that

$$\langle z, \bar{z} | n_+ n_- \rangle = \pi^{-\frac{1}{2}} \left( \frac{n_+!}{n_-!} \right)^{\frac{1}{2}} (-1)^{n_+} \bar{z}^{(n_- - n_+)} e^{-\frac{1}{2} z \bar{z}} L_{n_+}^{n_- - n_+}(z \bar{z}).$$

We summarize this in the following theorem:

**Theorem 2.** *We consider the states  $|n_+ n_- \rangle$  in the  $\{Q\}$ -representation with position coordinates  $(x, y) \in \mathbb{R}^2$ , and we write  $z = x + iy$ . Then*

$$\langle z, \bar{z} | n_+ n_- \rangle = \begin{cases} \pi^{-\frac{1}{2}} \left( \frac{n_-!}{n_+!} \right)^{\frac{1}{2}} (-1)^{n_-} z^{(n_+ - n_-)} e^{-\frac{1}{2} z \bar{z}} L_{n_-}^{n_+ - n_-}(z \bar{z}) & \text{if } n_+ \geq n_-, \text{ and} \\ \pi^{-\frac{1}{2}} \left( \frac{n_+!}{n_-!} \right)^{\frac{1}{2}} (-1)^{n_+} \bar{z}^{(n_- - n_+)} e^{-\frac{1}{2} z \bar{z}} L_{n_+}^{n_- - n_+}(z \bar{z}) & \text{if } n_+ \leq n_-. \end{cases}$$

## 5. COMPARISON WITH LG MODES OF PARAXIAL LIGHT

The time-harmonic field amplitudes of LG optical modes are expressed in cylindrical coordinates, with propagation coordinate  $z$  along the beam axis, as

$$\tilde{u}_{p,\ell}(\mathbf{r}) \propto e^{-i\Phi(\mathbf{r})} e^{-\frac{r^2}{w(z)^2}} \left( \frac{r\sqrt{2}}{w(z)} \right)^{|\ell|} L_p^{|\ell|} \left( \frac{2r^2}{w(z)^2} \right)$$

where the phase is

$$\Phi(\mathbf{r}) = \ell\phi - kz + \frac{kr^2}{2R(z)} - (2p + \ell + 1)\Psi(z)$$

and where again  $L_p^{|\ell|}$  is the Laguerre polynomial with radial and azimuthal indices  $p$  and  $\ell$ , respectively. The term  $\Psi(z) = \tan^{-1}(\frac{z}{z_R})$  is part of the Gouy phase,  $R(z) = \frac{z_R^2 + z^2}{z}$  is the radius of curvature,  $w(z) = w_0 \sqrt{1 + (\frac{z}{z_R})^2}$  is the beam waist, and  $z_R = \frac{1}{2} k w_0^2$  is the Rayleigh range. We see that the total phase evolves helically along the propagation coordinate  $z$ . For more on this, see, for example, [2], [3], [10].

In the transversal plane at the beam waist ( $z=0$ ), we get

$$\tilde{u}_{p,\ell}(\mathbf{r}) \propto e^{-i\ell\phi} e^{-\frac{r^2}{w_0^2}} \left( \frac{r\sqrt{2}}{w_0} \right)^{|\ell|} L_p^{|\ell|} \left( \frac{2r^2}{w_0^2} \right),$$

which is essentially what we have in Theorem 2.

## 6. LG MODES ARE WIGNER TRANSFORMS

Physicists in the optics community recently computed the Wigner transforms of LG modes [6], [7]. Surprisingly, the states  $|n_+ n_- \rangle$  are already themselves Wigner transforms of Hermite functions. The Hermite functions, as in Section 2, are given by

$$h_j(x) = \pi^{-\frac{1}{4}} (j!)^{-\frac{1}{2}} 2^{-\frac{j}{2}} (-1)^j e^{\frac{1}{2}x^2} \frac{d^j}{dx^j} e^{-x^2},$$

which, as we know, satisfy the orthogonality relations

$$\langle h_j | h_k \rangle = \delta_{jk}.$$

And we continue to write the HG modes as

$$h_{jk}(x, y) = h_j(x) h_k(y).$$

Now, for reference, we list the standard properties of the Wigner transform:

$$(4) \quad W(f, g) = \overline{W(g, f)},$$

$$(5) \quad \int W(f, g)(x, \xi) d\xi = \sqrt{2\pi} f\left(\frac{x}{\sqrt{2}}\right) g\left(\frac{x}{\sqrt{2}}\right),$$

$$(6) \quad \iint W(f, g)(x, \xi) dx d\xi = 2\sqrt{\pi} \langle f | g \rangle,$$

$$(7) \quad \int W(f, g)(x, \xi) dx = \sqrt{2\pi} \hat{f}\left(\frac{\xi}{\sqrt{2}}\right) \hat{g}\left(\frac{\xi}{\sqrt{2}}\right),$$

where  $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-ix\xi} f(x) dx$  is the Fourier transform, and (“Moyal’s identity”)

$$(8) \quad \langle W(f_1, g_1) | W(f_2, g_2) \rangle = \overline{\langle f_1 | f_2 \rangle} \langle g_1 | g_2 \rangle.$$

These properties are well known and are easily checked. But there is another elementary property which seems to be a new observation:

**Theorem 3.** *The extended Wigner transform intertwines the creation operators of the HG and LG modes, and it intertwines the annihilation operators of the HG and LG modes:*

$$\begin{aligned} A_+^\dagger \tilde{W} &= \tilde{W} a_1^\dagger, & A_-^\dagger \tilde{W} &= \tilde{W} a_2^\dagger, \\ A_+ \tilde{W} &= \tilde{W} a_1, & A_- \tilde{W} &= \tilde{W} a_2. \end{aligned}$$



The proof is a direct calculation, using the expressions (2) and (3). As a first application, it is now easy to see that the extended Wigner transform commutes with the operators  $H$  and  $N$ .

Additionally using the fact that  $\tilde{W}(h_{00}) = h_{00}$ , we have the following corollary:

**Corollary 1.** *We consider the LG modes  $|n_+n_- \rangle$  in the  $\{Q\}$ -representation with position coordinates  $(x, y) \in \mathbb{R}^2$ . Then we have*

$$\langle x, y | n_+ n_- \rangle = W(h_{n_+}, h_{n_-})(x, y).$$

Combining this with Theorem 2, we have a formula:

**Corollary 2.** *Suppose  $x, y \in \mathbb{R}$ , and let  $z = x + iy$ . Then*

$$W(h_j, h_k)(x, y) = \begin{cases} \pi^{-\frac{1}{2}} \left( \frac{k!}{j!} \right)^{\frac{1}{2}} (-1)^k z^{j-k} e^{-\frac{1}{2}z\bar{z}} L_k^{j-k}(z\bar{z}) & \text{if } j \geq k, \text{ and} \\ \pi^{-\frac{1}{2}} \left( \frac{j!}{k!} \right)^{\frac{1}{2}} (-1)^j \bar{z}^{k-j} e^{-\frac{1}{2}z\bar{z}} L_j^{k-j}(z\bar{z}) & \text{if } j \leq k. \end{cases}$$

Corollary 2 by itself is not new. For example, two independent proofs may be found in Folland's book [8], although we caution the reader that his statement of the result contains a misprint.

So we have yet another way of understanding the orthogonality relations of the states  $|n_+n_- \rangle$ : as being inherited from the orthogonality of the Hermite functions. Explicitly,

$$\begin{aligned} \langle m_+ m_- | n_+ n_- \rangle &= \langle W(h_{m_+}, h_{m_-}) | W(h_{n_+}, h_{n_-}) \rangle \\ &= \langle h_{m_+} | h_{n_+} \rangle \langle h_{m_-} | h_{n_-} \rangle \\ &= \delta_{m_+ n_+} \delta_{m_- n_-}. \end{aligned}$$

As another application of Corollary 1, we can think of the LG modes  $|n_+n_- \rangle$  as being “interference terms” resulting from the quadratic nature of the Wigner transform. If we apply the Wigner transform to sums of Hermite functions, we get interference terms that are LG modes with nonzero angular momentum. In fact, we have the following polarization identity:

$$\begin{aligned} \langle x, y | n_+ n_- \rangle &= \frac{1}{4} W(h_{n_+} + h_{n_-})(x, y) - \frac{1}{4} W(h_{n_+} - h_{n_-})(x, y) \\ &\quad + \frac{i}{4} W(h_{n_+} - ih_{n_-})(x, y) - \frac{i}{4} W(h_{n_+} + ih_{n_-})(x, y). \end{aligned}$$

One may think of the Wigner transform as being a kind of “product”. Just as Hermite functions give rise to two-dimensional HG modes, via the tensor product, Hermite functions also give rise to two-dimensional LG modes, via the Wigner transform. To complete this series of relationships, Simon and Agarwal [7] noted that the two-dimensional HG

and LG modes are unitarily related by the operator

$$\exp\left(\frac{i\pi}{4}\hat{T}_1\right), \quad \text{where } \hat{T}_1 = xy - \frac{\partial^2}{\partial x \partial y}.$$

Alternatively, we can use the Wigner transform to again show that the HG and LG modes are unitarily related. For functions  $F$  of two variables, we consider the extended Wigner transform of  $F$ :

$$\tilde{W}F(x, y) = \frac{1}{\sqrt{2\pi}} \int e^{ipy} F\left(\frac{x+p}{\sqrt{2}}, \frac{x-p}{\sqrt{2}}\right) dp.$$

If we write the  $\frac{\pi}{4}$ -rotation operator as

$$\begin{aligned} (R_{\frac{\pi}{4}}^* F)(x, p) &= F(R_{\frac{\pi}{4}}(x, p)) \\ &= F\left(\frac{x-p}{\sqrt{2}}, \frac{x+p}{\sqrt{2}}\right), \end{aligned}$$

and if we write the partial Fourier transform as

$$(\mathcal{F}_2 F)(x, y) = \frac{1}{\sqrt{2\pi}} \int e^{-ipy} F(x, p) dp,$$

then we may write the extended Wigner transform as

$$(9) \quad \tilde{W}F(x, y) = \left(\mathcal{F}_2 R_{\frac{\pi}{4}}^* F\right)(x, y).$$

This is clearly a unitary operator, so, in particular, we again see that the HG modes  $h_{jk}$  are unitarily related to the LG modes  $\tilde{W}(h_{jk})$ .

The results of this section are surprising, despite their mathematical simplicity. We are accustomed to thinking of the Wigner transform of a function as being a phase space representation of the function, but here the relevant Wigner transform is actually a function of the position vector  $(x, y) \in \mathbb{R}^2$ , as it is an LG mode. So if we use the Wigner transform to study the phase space properties of LG modes, we are really taking Wigner transforms of Wigner transforms. Gase [6], followed by Simon and Agarwal [7], already recently derived expressions for Wigner transforms of LG modes (Gase in terms of a quadruple sum, and Simon and Agarwal in terms of a closed form involving no summation at all). We will do the same in the next section, but now using our new point of view.

## 7. WIGNER TRANSFORMS OF LG MODES

We now proceed to compute the Wigner transforms of the LG modes  $\tilde{W}(h_{jk})$ . In light of (9), the first step is to consider the action of the Wigner transform on partial Fourier transforms.

**Lemma 1.**

$$W_2(\mathcal{F}_2 F, \mathcal{F}_2 G)(\vec{x}, \vec{\xi}) = \frac{1}{2\pi} \iint e^{i\xi_1 s - i x_2 t} \overline{F\left(\frac{x_1 + s}{\sqrt{2}}, \frac{-\xi_2 - t}{\sqrt{2}}\right)} G\left(\frac{x_1 - s}{\sqrt{2}}, \frac{-\xi_2 + t}{\sqrt{2}}\right) ds dt.$$

This lemma follows from a direct calculation.

We now apply the lemma to

$$F = R_{\frac{\pi}{4}}^* h_{jk} \quad \text{and} \quad G = R_{\frac{\pi}{4}}^* h_{mn}.$$

That is,

$$F(\vec{x}) = h_j \left( \frac{x_1 - x_2}{\sqrt{2}} \right) h_k \left( \frac{x_1 + x_2}{\sqrt{2}} \right)$$

and similarly for  $G$ . Hence the Wigner transforms of the LG modes are given by

$$\begin{aligned} W_2(\tilde{W}(h_{jk}), \tilde{W}(h_{mn}))(\vec{x}, \vec{\xi}) \\ = \frac{1}{2\pi} \iint e^{i\xi_1 s - i x_2 t} h_j \left( \frac{(x_1 + s) + (\xi_2 + t)}{2} \right) h_k \left( \frac{(x_1 + s) - (\xi_2 + t)}{2} \right) \\ h_m \left( \frac{(x_1 - s) + (\xi_2 - t)}{2} \right) h_n \left( \frac{(x_1 - s) - (\xi_2 - t)}{2} \right) ds dt. \end{aligned}$$

We may simplify this integral by again rotating the variables by  $\frac{\pi}{4}$ . We state the result as a theorem.

**Theorem 4.** *The Wigner transforms of the LG modes are products of LG modes, given by the formula*

$$\begin{aligned} W_2(\tilde{W}(h_{jk}), \tilde{W}(h_{mn}))(\vec{x}, \vec{\xi}) \\ = \tilde{W}(h_{jm}) \left( \frac{x_1 + \xi_2}{\sqrt{2}}, \frac{\xi_1 - x_2}{\sqrt{2}} \right) \tilde{W}(h_{kn}) \left( \frac{x_1 - \xi_2}{\sqrt{2}}, \frac{\xi_1 + x_2}{\sqrt{2}} \right). \end{aligned}$$

In the special case  $(j, k) = (m, n)$ , we have the following formula of Simon and Agarwal [7]. It is now proven by combining Corollary 2 with Theorem 4.

**Corollary 3.** *For  $(\vec{x}, \vec{\xi}) = (x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ , we let*

$$Q_0 = \frac{1}{2}(|\vec{x}|^2 + |\vec{\xi}|^2) \quad \text{and} \quad Q_2 = (x_1 \xi_2 - x_2 \xi_1).$$

*Then we have*

$$W_2(\tilde{W}(h_{jk}), \tilde{W}(h_{jk}))(\vec{x}, \vec{\xi}) = \pi^{-1} (-1)^{j+k} e^{-Q_0} L_j^0(Q_0 + Q_2) L_k^0(Q_0 - Q_2).$$

For comparison, we also give the Wigner transforms of HG modes:

$$W_2(h_{jk}, h_{mn})(\vec{x}, \vec{\xi}) = \tilde{W}(h_{jm})(x_1, \xi_1) \tilde{W}(h_{kn})(x_2, \xi_2).$$

In the particular case  $(j, k) = (m, n)$ , if we let

$$Q_3 = \frac{1}{2}(x_1^2 - x_2^2 + \xi_1^2 - \xi_2^2),$$

then we have the formula

$$W_2(h_{jk}, h_{jk})(\vec{x}, \vec{\xi}) = \pi^{-1} (-1)^{j+k} e^{-Q_0} L_j^0(Q_0 + Q_3) L_k^0(Q_0 - Q_3).$$

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